ON A CERTAIN PROPERTY OF THE CHARACTERISTIC NUMBERS OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS

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This paper will study a property of the characteristic numbers of the vanishing solutions of the equations of disturbed motion. A theorem will be established on a very close link between the characteristic numbers of the above solutions and the eigenvalues of the system of first-order approximations. It is difficult to establish a condition for the stability of such characteristic numbers, which has been used earlier for the proof of an analogous theorem [2].

Let there be given the system of equations of disturbed motion

$$\frac{dx_1}{dt} = X_1, \ldots, \qquad \frac{dx_n}{dt} = X_n \tag{1}$$

where $X_1, \ldots, X_n$ are the holomorphic functions of $x_1, \ldots, x_n$,

$$X_s = p_{s1} x_1 + \ldots + p_{sn} x_n + \sum P_s^{(m_1, \ldots, m_n)} x_1^{m_1} \ldots x_n^{m_n},$$

with the sum extended over all non-negative integers $m_1, \ldots, m_n$, if $m_1 + \ldots + m_n > 1$.

The coefficients p_{si}, $P_s^{(m_1, \ldots, m_n)}$ are real, continuous, bounded functions of time, and there exist positive constants M and A such that for all $t \geqslant t_0$

$$|P_s^{(m_1, \ldots, m_n)}| < \frac{M}{A^{m_1 + \ldots + m_n}} \tag{2}$$

Consider the system of first approximations

$$\frac{dx_s}{dt} = p_{s1} x_1 + \ldots + p_{sn} x_n \qquad (s = 1, \ldots, n) \tag{3}$$

and its characteristic numbers $\lambda_1, \ldots, \lambda_n$.

In reference [2], the following theorem has been established:

Theorem: If the system (1) has a vanishing solution and the system (3) stable eigenvalues, the characteristic number of this solution is exactly equal to one of the nonnegative eigenvalues of the system (3).

This theorem can be proved without condition (2). However, if this condition is added and if, in addition, it is assumed that the system (3) is regular and that among its eigenvalues we have $\lambda_1 \geqslant \lambda_2 \quad \ldots \geqslant \lambda_p > 0$, a stronger result may be established, even if we relax the condition of the stability of the eigenvalues, which is not readily verified.

Theorem: *a*) If the system (3) is regular and $\lambda_1 \geqslant \lambda_2 \quad \ldots \geqslant \lambda_k > 0$, system (1) must have vanishing solutions $x_1{}^j, \ldots, x_n{}^j$ with characteristic numbers $\lambda_1, \ldots, \lambda_k$.

b) For a solution with the initial conditions $x_1{}^0, \ldots, x_n{}^0$ to have the characteristic number $\lambda_1 > \lambda_p$, it is sufficient to fix $x_1{}^0, \ldots, x_p{}^0$ arbitrarily, except for ensuring that their moduli are sufficiently small, and to find $x_{p+1}{}^0, \ldots, x_n{}^0$ from the relations

$$x_{p+1} = \varphi_{p+1}(x_1{}^\circ, \ldots, x_p{}^\circ), \qquad x_n = \varphi_n(x_1{}^\circ, \ldots, x_p{}^\circ)$$

where $\phi_{p+1}, \ldots, \phi_n$ are holomorphic functions of $x_1{}^0, \ldots, x_p{}^0$ which vanish when the latter vanish.

Before presenting the proof, two theorems of Liapunov's first method will be stated.

Let $x_{ij}, \ldots, x_{nj}$ be a normal system of independent solutions of the system of equations (3).

Liapunov's theorem: I. If $\lambda_1, \ldots, \lambda_k$ are positive eigenvalues of the regular system (3), system (1) has solutions which may be presented in the form

$$x_s = \sum_{j=1}^{k} \alpha_j x_{js} + \sum_{m_1+\ldots+m_k>2} L_s^{(m_1, \ldots, m_k)} \alpha_1{}^{m_1} \ldots \alpha_k{}^{m_k} \exp\Big(-\sum_{i=1}^{k} m_i \lambda_i t\Big) \qquad (4)$$

where $\alpha_1, \ldots, \alpha_k$ are arbitrary constants whose moduli do not exceed some upper nonzero-bound and the eigenvalues of the functions $L_s^{(m_1, \ldots, m_n)}$ are not less than zero.

II. If ϵ is some positive constant and one puts

$$\alpha_s e^{-(\lambda_s - \varepsilon)t} = q_s \qquad (s = 1, \ldots, k)$$

and replaces the α_s in the series (4) by the corresponding expressions, one obtains the new series

$$x_s = \sum Q_s^{(m_1, \ldots, m_k)} q_1^{m_1} \ldots q_k^{m_k} \qquad (s = 1, \ldots, n)$$

expanded in ascending powers of q_s which will have the property that for every ϵ, however small, there will be a positive constant $Q^{(m_1, \ldots, m_k)}$ for which, for all nonnegative values of t, one has the inequalities

$$|Q_s^{(m_1, \ldots, m_k)}| < Q^{(m_1, \ldots, m_k)}$$

The series

$$\sum Q^{(m_1, \ldots, m_k)} q_1^{m_1}, \ldots, q_k^{m_k}$$

will converge absolutely, as long as the quantities q_s do not exceed some nonzero-bound q. It will be noted that a majorant for the functions

$$Q_s^{(m_1, \ldots, m_k)} = L_s^{(m_1, \ldots, m_k)} e^{-(m_1 + \ldots + m_k)\varepsilon t}$$

may be constructed for any $\epsilon > 0$, however small.

Proceeding now to the proof of the theorems, consider the solution

$$x_s^p = \sum_{j=1}^{p} \alpha_j x_{js} + \sum_{1 < m_1 + \ldots + m_p < l} L^{(m_1 \ldots m_p)} \alpha_1^{m_1} \ldots \alpha_p^{m_p} \exp\left(-\sum_{i=1}^{p} m_i \lambda_i t\right) +$$
$$+ \sum_{\infty > m_1 + \ldots + m_p > l} L_s^{(m_1, \ldots, m_p)} \alpha_1^{m_1} \ldots \alpha_p^{m_p} \left(\exp - \sum_{i=1}^{p} m_i \lambda_i t\right)$$

in which $\alpha_{p+1}, \ldots, \alpha_k$ have been put equal to zero, and the second sums on the right-hand sides contain all terms of order smaller than l in relation to the constants $\alpha_1, \ldots, \alpha_p$. For this purpose, l is some positive integer, greater than 4, which may be fixed arbitrarily.

On the basis of the fact that

$$X\,(L_s^{(m_1, \ldots, m_p)}) \geqslant 0$$

and of a theorem involving the sums and products of eigenvalues, it is readily verified that for any $\alpha_1, \ldots, \alpha_p$, not all simultaneously zero,

$$X\left\{\sum_{1 < m_1 + \ldots + m_p < l} L_s^{(m_1, \ldots, m_p)} \exp\left(-\sum_{i=1}^{p} m_i \lambda_i t\right)\right\} \geqslant 2\lambda_p$$

$$X\left\{\sum_{j=1}^{p} \alpha_j x_{js}\right\} = \lambda_i \geqslant \lambda_p$$

where λ_s is one of the numbers $\lambda_1, \ldots, \lambda_p$.

The symbol $X\{f_s\}$ denotes the eigenvalue of the function f_s of the system.

Consideration will now be given to the eigenvalue of the last sum and it will be shown that it is greater than λ_p. For this purpose it will be sufficient to find a small $\delta > 0$ such that the functions

$$e^{(\lambda_p+\delta)t} \sum_{\infty \geqslant m_1+\ldots+m_p \geqslant l} L_s^{(m_1,\ldots,m_p)} \alpha_1^{m_1} \ldots \alpha_p^{m_p} \exp\left[-\sum_{i=1}^{p} m_i \lambda_i t\right] \qquad (s = 1, 2, \ldots, n) \tag{5}$$

are bounded.

They may be written in the form

$$\sum_{\infty \geqslant m_1+\ldots+m_p \geqslant l} L_s^{(m_1,\ldots,m_p)} \alpha_1^{m_1} \ldots \alpha_p^{m_p} \exp\left(-\sum_{i=1}^{p} \lambda_i m_i - \lambda_p - \delta\right) t$$

and subjected to the transformation:

$$q_1 = \alpha_1 \exp\left[-\left(\lambda_1 - \frac{1}{3}\lambda_p\right)t\right], \qquad q_p = \alpha_p \exp\left[-\left(\lambda_p - \frac{1}{3}\lambda_p\right)t\right]$$

Then the series subsequently obtained will have in the capacity of $Q_s^{(m_1, \ldots, m_p)}$ the quantities

$$L_s^{(m_1,\ldots,m_p)} \exp - \sum \left(m_i \frac{\lambda_p}{3} - \lambda_p - \delta\right) t$$

Putting $\delta = 1/3\, \lambda_p$, we then find

$$\left| L_s^{(m_1,\ldots,m_p)} \exp - \sum (m_i - 4) \frac{\lambda_p}{3} t \right| < | L_s^{(m_1,\ldots,m_p)} \exp - (m_1 + \ldots + m_p) \varepsilon t \tag{6}$$

where $\epsilon > 0$ must be chosen in the following manner.

Since

$$-\sum (m_i - 4) \frac{1}{3}\lambda_p \leqslant -(l-4)\frac{1}{3}\lambda_p$$

we obtain, by fixing ϵ to fulfil the inequality

$$(l-4)\, {}^1\!/_3 \lambda_p > l\varepsilon, \quad \text{or} \quad 0 < \varepsilon < \left(1 - \frac{4}{l}\right) {}^1\!/_3 \lambda_p$$

that for any $m_1 + \ldots + m_p > l$ the inequality

$$-\sum (m_i - 4) \frac{\lambda_p}{3} < -(m_1 + \ldots + m_p)\varepsilon$$

will be satisfied together with (5).

Since the series obtained from the substitutions

$$q_s = \alpha_s \exp - (\lambda_s - \varepsilon)\, t \qquad (s = 1, \ldots, p)$$

will have as majorants the absolutely convergent series

$$\sum_{l \leqslant m_1 + \ldots + m_p \leqslant \infty} Q^{(m_1, \ldots, m_p)}\, q_1^{m_1} \ldots q_p^{m_p}$$

the series studied (5), on the strength of the inequalities (6), will likewise have this property. It is clear from the form of the substitution used to obtain them, that for any small $|a_1|, \ldots, |a_p|$ all the series (5) must be vanishing functions of the time, and hence they must be bounded.

Now putting $a_1 = \ldots = a_p = 0$ and $a_p \neq 0$, we obtain

$$(x_s^p)^\circ = \alpha_p x_{ps} + \sum_{1 < m_p < l} L_s^{(m_p)} \alpha_p^{m_p} \exp(-m_p \lambda_p t) +$$

$$+ \sum_{l \leqslant m_p \leqslant \infty} L_s^{(m_p)} \alpha_p^{m_p} \exp - (m_p \lambda_p t)$$

Since it has been established that

$$X \Big\{ \sum_{1 < m_p < l} L_s^{(m_p)} \alpha_s^{m_p} \exp(-m_p \lambda_p t) \Big\} \geqslant 2\lambda_p$$

$$X \Big\{ \sum_{l \leqslant m_p \leqslant \infty} L_s^{(m_p)} \alpha_s^{m_p} \exp(-m_p \lambda_p t) \Big\} > \lambda_p$$

by the condition $X\{x_{ps}\} = \lambda_p$, then, on the basis of the theorem for the sum of eigenvalues, we may establish that

$${}^{d}\chi = \{_\circ({}_d{}^s x)\}\; X$$

which completes the proof of the first part of the theorem.

For the proof of its second part let $t = t_0$, when

$$x_s^{\circ p} = \alpha_1 x_{1s}^\circ + \ldots + \alpha_p x_{ps}^\circ + \psi_s(\alpha_1, \ldots, \alpha_p) \qquad (s = 1, \ldots, n) \tag{7}$$

where $\psi(a_1, \ldots, a_p)$ are holomorphic functions of $a_1, \ldots, a_p$ which begin with terms not smaller than second order in $a_1, \ldots, a_p$. Among the minors of the matrix $(x_{js}^{\,0})$ there will certainly be one different from zero, since otherwise the solutions x_{js} would not be linearly independent. Let this minor correspond to the first p rows of the matrix. Then the equations

$$\begin{aligned} x_1^{\circ p} &= \alpha_1 x_{11}^\circ + \ldots \alpha_p x_{1p}^\circ + \psi_1(\alpha_1, \ldots, \alpha_p) \\ x_p^{\circ p} &= \alpha_{1p} x_{1p}^\circ + \ldots \alpha_p x_{pp}^\circ + \psi_p(\alpha_1, \ldots \alpha_p) \end{aligned} \tag{8}$$

can be solved for $a_1, \ldots, a_p$, and the functions

$$\alpha_1 = X_1(x_1^{\circ p}, \ldots, x_p^{\circ p}), \qquad \alpha_p = X_p(x_1^{\circ p}, \ldots x_p^{\circ p}) \tag{9}$$

will be holomorphic functions of $x_1^{0p}, \ldots, x_p^{0p}$, if the moduli of the latter are sufficiently small, and certainly will not all vanish unless all $x_1^{0p}, \ldots, x_p^{0p}$ are zero.

For a given system of values of $x_1^{0p}, \ldots, x_p^{0p}$ let the $\alpha_1, \ldots, \alpha_p$, not all equal to zero, be found from the equations (9). Then, as has been shown above, the solution

$$x_s^p = \sum_{j=1}^{p} \alpha_j x_{js} + \sum_{1 < m_1 + \ldots + m_p \leqslant \infty} L_s^{(m_1 \ldots m_p)} \alpha_1^{m_1} \ldots \alpha_p^{m_p} \exp\left(-\sum_{i=1}^{p} m_i \lambda_i t\right)$$

will have eigenvalues which are not smaller than λ_p.

The remaining $x_{p+1}^0, \ldots, x_n^0$ of this solution may be found from equations (7). If it is desired to find them directly in terms of $x_1^{0p}, \ldots, x_p^{0p}$, then in the last $n - p$ equations (7), for $\alpha_1, \ldots, \alpha_p$ we must substitute their values (9). The relations obtained in this manner will play the part of the equations mentioned in the formulation of the theorem.

Note. Generally speaking, it does not follow from what has been proved that the eigenvalue of the solution (4) is exactly equal to one of the numbers $\lambda_1, \ldots, \lambda_p$.

The stated method offers the possibility of showing only that it is either equal to one of the numbers $\lambda_1, \ldots, \lambda_p$ or not smaller than $2\lambda_p$, if

$$2\lambda_p \leqslant \lambda_j, \qquad \lambda_j = X\left\{\sum_{j=1}^{p} \alpha_j x_{js}\right\}$$

If $2\lambda_p \neq \lambda_j$, $j = 1, \ldots, p - 1$, the last case certainly does not occur when $\lambda_1, \ldots, \lambda_p$ are stable eigenvalues. In fact, by assuming the opposite and verifying that the solution $x_{ij}(t)$ vanishes, we may arrive at a contradiction to the theorem proved in [2].

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